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ZEROS OF A CERTAIN CLASS OF POLYNOMIALS

M.H. Gulzar*, B.A.Zargar, A.W.Manzoor
* Department of Mathematics, University of Kashmir, Srinagar

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ABSTRACT

In this paper we locate the regions containing all or some of the zeros of a certain class of polynomials subjected to certain coefficient conditions.

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KEYWORDS: Coefficients, Polynomial, Zeros.

INTRODUCTION

The famous Theorem of Enestrom-Kakeya [9] states that all the zeros of a polynomial $P(z)=\sum_{j=0}^{n} a_j z^j$ satisfying $a_n \ge a_{n-1} \ge .... \ge a_1 \ge a_0 > 0$ lie in $|z| \le 1$. In the literature there exist several generalizations and refinements of this theorem[1-11]. Very recently Gulzar et al [7] proved the following result:

Theorem A: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge \tau a_\lambda$$

and

$$L = |a_\lambda - a_{\lambda-1}| + |a_{\lambda-1} - a_{\lambda-2}| + ... + |a_1 - a_0| + |a_0|.$$

Then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{ka_n - \tau a_\lambda + (1-\tau)|a_\lambda| + L}{|a_n|}.$$

Theorem B: Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda, 0 \le \lambda \le n-1$ and for some $k \ge 1, o < \tau \le 1$,

$$k\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_{\lambda+1} \ge \tau\alpha_\lambda$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|.$$

Then the number f zeros of P(z) in $|z| \le \delta, 0 < \delta < 1$ does not exceed

$$\frac{1}{\log\frac{1}{\delta}}\log\frac{|a_n|+(k-1)|\alpha_n|+k\alpha_n-\tau\alpha_\lambda+L+(1-\tau)|\alpha_\lambda|+2\sum_{j=0}^{n}|\beta_j|}{|a_0|}.$$

MAIN RESULTS

In this paper we prove the following results:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu,0\le\lambda\le n-1,0<\mu\le 1$ and for some $k_1,k_2\ge 1;0<\tau_1,\tau_2\le 1,$,

$$k_1\alpha_n \ge \alpha_{n-1}\ge ......\ge \tau_1\alpha_\lambda,$$
$$k_2\beta_n \ge \beta_{n-1}\ge ......\ge \tau_2\beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}[k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(\alpha_\lambda+|\alpha_\lambda|)$$

.

$$-\tau_2(\beta_\mu+|\beta_\mu|)]$$

Remark 1: If a_j is real i.e. $\beta_j=0,\forall j=0,1,......,n;k_1=k,\tau_1=\tau$, Theorem 1 reduces to Theorem A.

Taking $\tau_1=\tau_2=1$, Theorem 1 gives the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu,0\le\lambda\le n-1,0<\mu\le 1$ and for some $k_1,k_2\ge 1,$,

$$k_1\alpha_n \ge \alpha_{n-1}\ge ......\ge \alpha_\lambda,$$
$$k_2\beta_n \ge \beta_{n-1}\ge ......\ge \beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{(k-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}[k_1\alpha_n+k_2\beta_n+L+M-\alpha_\gamma-\beta_\mu].$$

Taking $k_1=k_2=\tau_1=\tau_2=1$, Theorem 1 gives the following result:

Corollary 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu, 0\le\lambda\le n-1, 0<\mu\le 1$,

$$\alpha_n \ge \alpha_{n-1} \ge ...... \ge \alpha_\lambda,$$
$$\beta_n \ge \beta_{n-1} \ge ...... \ge \beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then all the zeros of P(z) lie in

$$|z|\le\frac{1}{|a_n|}[\alpha_n+\beta_n+L+M-\alpha_\lambda-\beta_\mu].$$

Next, we prove the following result:

Theorem 2: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu, 0\le\lambda\le n-1, 0<\mu\le 1$ and for some $k_1,k_2\ge 1; 0<\tau_1,\tau_2\le 1,$,

$$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge \tau_1\alpha_\lambda,$$
$$k_2\beta_n \ge \beta_{n-1} \ge ...... \ge \tau_2\beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{K}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|a_0|}$ for $R\ge 1$ and the number of zeros of P(z) in $\frac{|a_0|}{K'}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{M'}{|a_0|}$ for $R\le 1$, where

$$K=|a_n|R^{n+1}+R^n\{(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_0|-|\beta_0|\},$$
$$K'=|a_n|R^{n+1}+R\{(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda))$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_0|-|\beta_0|\},$$
$$M=|a_n|R^{n+1}+R^n[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)],$$

$$M'=|a_n|R^{n+1}+R[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$

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$$-\tau_2(|\beta_\mu|+\beta_\mu)]+(1-R)(|\alpha_0|+|\beta_0|) \ .$$

Taking $\tau_1=\tau_2=1$, Theorem 2 gives the following result:

Corollary 3: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu, 0\le\lambda\le n-1, 0<\mu\le 1$ and for some $k_1,k_2\ge 1;$,

$$k_1\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_\lambda,$$
$$k_2\beta_n\ge\beta_{n-1}\ge......\ge\beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{K}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|a_0|}$ for $R\ge1$ and the number of zeros of P(z) in $\frac{|a_0|}{K'}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{M'}{|a_0|}$ for $R\le1$, where

$$K=|a_n|R^{n+1}+R^n\{(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+L+M-\alpha_\lambda-\beta_\mu-|\alpha_0|-|\beta_0|\},$$

$$K'=|a_n|R^{n+1}+R\{(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+L+M-\alpha_\lambda-\beta_\mu-|\alpha_0|-|\beta_0|\},$$
$$M=|a_n|R^{n+1}+R^n[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+L+M-\alpha_\lambda-\beta_\mu] \qquad ,$$
$$M'=|a_n|R^{n+1}+R[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+L+M-\alpha_\lambda-\beta_\mu]+(1-R)(|\alpha_0|+|\beta_0|)$$
.

Taking $k_1=k_2=\tau_1=\tau_2=1$, Theorem 2 gives the following result:

Corollary 4: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j)=\alpha_j, \text{Im}(a_j)=\beta_j, j=0,1,2,......,n$ such that for some $\lambda,\mu, 0\le\lambda\le n-1, 0<\mu\le 1$,

$$\alpha_n\ge\alpha_{n-1}\ge......\ge\alpha_\lambda,$$
$$\beta_n\ge\beta_{n-1}\ge......\ge\beta_\mu,$$

and

$$L=|\alpha_\lambda-\alpha_{\lambda-1}|+|\alpha_{\lambda-1}-\alpha_{\lambda-2}|+......+|\alpha_1-\alpha_0|+|\alpha_0|,$$
$$M=|\beta_\mu-\beta_{\mu-1}|+|\beta_{\mu-1}-\beta_{\mu-2}|+......+|\beta_1-\beta_0|+|\beta_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{K}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|a_0|}$ for $R\ge1$ and the number of zeros of P(z) in $\frac{|a_0|}{K'}\le|z|\le\frac{R}{c}, c>1$ is less than or equal to

$\frac{1}{\log c}\log\frac{M'}{|a_0|}$ for $R \le 1$, where

$K = |a_n|R^{n+1} + R^n\{\alpha_n + \beta_n + L + M - \alpha_\lambda - \beta_\mu - |\alpha_0| - |\beta_0|\}$,

$K' = |a_n|R^{n+1} + R\{\alpha_n + \beta_n + L + M - \alpha_\lambda - \beta_\mu - |\alpha_0| - |\beta_0|\}$,

$M = |a_n|R^{n+1} + R^n[\alpha_n + \beta_n + L + M - \alpha_\lambda - \beta_\mu]$,

$M' = |a_n|R^{n+1} + R[\alpha_n + \beta_n + L + M - \alpha_\lambda - \beta_\mu] + (1-R)(|\alpha_0| + |\beta_0|)$.

Taking R=1 and $c = \frac{1}{\delta}, 0 < \delta < 1$ in Theorem 2, we get the following result:

Corollary 5: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j, j = 0,1,2,......,n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 < \mu \le 1$ and for some $k_1, k_2 \ge 1; 0 < \tau_1, \tau_2 \le 1$,

$$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge \tau_1\alpha_\lambda,$$
$$k_2\beta_n \ge \beta_{n-1} \ge ...... \ge \tau_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$
$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0|.$$

Then the number of zeros of P(z) in $\frac{|a_0|}{K} \le |z| \le \delta, 0 < \delta < 1$ is less than or equal to

$\frac{1}{\log\frac{1}{\delta}}\log\frac{M}{|a_0|}$, where

$$K = |a_n| + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$- \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|,$$
$$M = |a_n| + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$- \tau_2(|\beta_\mu| + \beta_\mu).$$

In particular for $\delta = \frac{1}{2}$, Cor.5 gives the following result:

Corollary 6: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with $\text{Re}(a_j) = \alpha_j, \text{Im}(a_j) = \beta_j, j = 0,1,2,......,n$ such that for some $\lambda, \mu, 0 \le \lambda \le n-1, 0 < \mu \le 1$ and for some $k_1, k_2 \ge 1; 0 < \tau_1, \tau_2 \le 1$,

$$k_1\alpha_n \ge \alpha_{n-1} \ge ...... \ge \tau_1\alpha_\lambda,$$
$$k_2\beta_n \ge \beta_{n-1} \ge ...... \ge \tau_2\beta_\mu,$$

and

$$L = |\alpha_\lambda - \alpha_{\lambda-1}| + |\alpha_{\lambda-1} - \alpha_{\lambda-2}| + ... + |\alpha_1 - \alpha_0| + |\alpha_0|,$$

$$M = |\beta_\mu - \beta_{\mu-1}| + |\beta_{\mu-1} - \beta_{\mu-2}| + ... + |\beta_1 - \beta_0| + |\beta_0| .$$

Then the number of zeros of P(z) in $\frac{|a_0|}{K} \le |z| \le \frac{1}{2}$ is less than or equal to

$\frac{1}{\log 2} \log \frac{M}{|a_0|}$, where

$$K = |a_n| + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$- \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0| ,$$
$$M = |a_n| + (k_1 - 1)|\alpha_n| + (k_2 - 1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$
$$- \tau_2(|\beta_\mu| + \beta_\mu) .$$

For other different choices of the parameters, we get many other interesting results.

Lemmas

For the proof of Theorem 2, we make use of the following lemmas:

Lemma 1: Let f(z) (not identically zero) be analytic for $|z| \le R,\ f(0) \ne 0$ and $f(a_k) = 0,\quad k = 1,2,......, n$. Then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(\mathrm{R}e^{i\theta}|d\theta - \log|f(0)| = \sum_{j=1}^{n} \log\frac{R}{|a_j|} .$$

Lemma 1 is the famous Jensen's Theorem (see page 208 of [1]).

Lemma 2: Let f(z) be analytic , $f(0) \ne 0$ and $|f(z)| \le M$ for $|z| \le R$. Then the number of zeros of f(z) in $|z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|f(0)|}$.

Lemma 2 is a simple deduction from Lemma 1.

PROOFS OF THEOREMS

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}
F(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + ... + (a_{\lambda+1} - a_\lambda)z^{\lambda+1} + (a_\lambda - a_{\lambda-1})z^\lambda \\
&\quad + ... + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} - (k_1 - 1)\alpha_n z^n + (k_1\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} ...... + (\alpha_{\lambda+1} - \tau_1\alpha_\lambda)z^{\lambda+1} \\
&\quad + (\tau_1 - 1)\alpha_\lambda z^{\lambda+1} + (\alpha_\lambda - \alpha_{\lambda-1})z^\lambda + ... + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(k_2\beta_n - \beta_{n-1})z^n \\
&\quad - (k_2 - 1)\beta_n z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} + ... + (\beta_{\mu+1} - \tau_2\beta_\mu)z^{\mu+1} + (\tau_2 - 1)\beta_\mu z^{\mu+1} \\
&\quad + (\beta_\mu - \beta_{\mu-1})z^\mu + ... + (\beta_1 - \beta_0)z + \beta_0\}
\end{aligned}$$

For $|z| > 1$ so that $\frac{1}{|z|^j} < 1, \forall j = 1,2,......, n$, we have, by using the hypothesis

$$|F(z)| \ge |a_n z + (k_1 - 1)\alpha_n + i(k_2 - 1)\beta_n||z|^n - [|k_1\alpha_n - \alpha_{n-1}||z|^n + |\alpha_{n-1} - \alpha_{n-2}|.|z|^{n-1} ..... + |\alpha_{\lambda+1} - \tau_1\alpha_\lambda||z|^{\lambda+1}$$

$$+|\tau_1-1||\alpha_\lambda||z|^{\lambda+1}+|\alpha_\lambda-\alpha_{\lambda-1}||z|^{\lambda}+......+|\alpha_1-\alpha_0||z|+|\alpha_0|+\{|k_2\beta_n-\beta_{n-1}||z|^n$$
$$+|\beta_{n-1}-\beta_{n-2}||z|^{n-1}+......+|\beta_{\mu+1}-\tau_2\beta_\mu||z|^{\mu+1}+|\beta_{\mu-1}-\beta_{\mu-2}||z|^{\mu}+......$$
$$+(|\beta_1-\beta_0||z|+|\beta_0|]$$

$$=|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|-\{|k_1\alpha_n-\alpha_{n-1}|+\frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|}+......$$
$$+\frac{|\alpha_{\lambda+1}-\tau_1\alpha_\lambda|}{|z|^{n-\lambda-1}}+\frac{(1-\tau_1)|\alpha_\gamma|}{|z|^{n-\lambda-1}}+\frac{|\alpha_\lambda-\alpha_{\lambda-1}|}{|z|^{n-\lambda}}+......+\frac{|\alpha_1-\alpha_0|}{|z|^{n-1}}+\frac{|\alpha_0|}{|z|^n}$$
$$+|k_2\beta_n-\tau_2\beta_{n-1}|+\frac{|\beta_{n-1}-\beta_{n-2}|}{|z|}+......+\frac{|\beta_{\mu+1}-\tau_2\beta_\mu|}{|z|^{n-\mu-1}}+\frac{|\beta_\mu-\tau\beta_{\mu-1}|}{|z|^{n-\mu}}$$
$$+......+\frac{|\beta_1-\beta_0|}{|z|^{n-1}}+\frac{|\beta_0|}{|z|^n}\}]$$

$$>|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|-\{|k_1\alpha_n-\alpha_{n-1}|+|\alpha_{n-1}-\alpha_{n-2}|+......$$
$$+|\alpha_{\lambda+1}-\tau\alpha_\lambda|+(1-\tau_1)|\alpha_\lambda|+|\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-\alpha_0|+|\alpha_0|$$
$$+|k_2\beta_n-\beta_{n-1}|+|\beta_{n-1}-\beta_{n-2}|+......+|\beta_{\mu+1}-\tau_2\beta_\mu|+(1-\tau_2)|\beta_\mu|+|\beta_\mu-\beta_{\mu-1}|$$
$$+......+|\beta_1-\beta_0|+|\beta_0|\}]$$

$$=|z|^n[|a_nz+(k_1-1)\alpha_n|-\{k_1\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+\alpha_{\lambda+1}-\tau_1\alpha_\lambda$$
$$+(1-\tau_1)|\alpha_\lambda|+|\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-\alpha_0|+|\alpha_0|+k_2\beta_n-\beta_{n-1}+\beta_{n-1}$$
$$-\beta_{n-2}+......+\beta_{\mu+1}-\tau_2\beta_\mu+(1-\tau_2)|\beta_\mu|+|\beta_\mu-\beta_{\mu-1}|......+|\beta_1-\beta_0|+|\beta_0|$$

$$\geq|z|^n[|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|-\{k_1\alpha_n+|\alpha_\lambda|+L-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$+k_2\beta_n+|\beta_\mu|+M-\tau_2(|\beta_\mu|+\beta_\mu\}]$$
$$>0$$

if

$$|a_nz+(k_1-1)\alpha_n+i(k_2-1)\beta_n|>k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)$$

i.e. if

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|>\frac{1}{|a_n|}[k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)].$$

This shows that those zeros of F(z) whose modulus is greater than 1 lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\leq\frac{1}{|a_n|}[k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)].$$

Since the zeros of F(z) whose modulus is less than or equal to 1 already satisfy the above inequality and since the zeros of P(z) are also the zeros of F(z) , it follows that all the zeros of P(z) lie in

$$\left|z+\frac{(k_1-1)\alpha_n+i(k_2-1)\beta_n}{a_n}\right|\le\frac{1}{|a_n|}[k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M-\tau_1(|\alpha_\lambda|+\alpha_\lambda)$$
$$-\tau_2(|\beta_\mu|+\beta_\mu)].$$

That proves Theorem 1.

Proof of Theorem 2: Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)(a_nz^n+a_{n-1}z^{n-1}+......+a_1z+a_0)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+......+(a_{\lambda+1}-a_\lambda)z^{\lambda+1}+(a_\lambda-a_{\lambda-1})z^\lambda\\&\quad+......+(a_1-a_0)z+a_0\\&=-a_nz^{n+1}-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}......+(\alpha_{\lambda+1}-\tau_1\alpha_\lambda)z^{\lambda+1}\\&\quad+(\tau_1-1)\alpha_\lambda z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+......+(\alpha_1-\alpha_0)z+i\{(k_2\beta_n-\beta_{n-1})z^n\\&\quad-(k_2-1)\beta_nz^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......+(\beta_{\mu+1}-\tau_2\beta_\mu)z^{\mu+1}+(\tau_2-1)\beta_\mu z^{\mu+1}\\&\quad+(\beta_\mu-\beta_{\mu-1})z^\mu+......+(\beta_1-\beta_0)z\}+a_0\\&=G(z)+a_0,\end{aligned}$$

where

$$\begin{aligned}G(z)=&-a_nz^{n+1}-(k_1-1)\alpha_nz^n+(k_1\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}......+(\alpha_{\lambda+1}-\tau_1\alpha_\lambda)z^{\lambda+1}\\&+(\tau_1-1)\alpha_\lambda z^{\lambda+1}+(\alpha_\lambda-\alpha_{\lambda-1})z^\lambda+......+(\alpha_1-\alpha_0)z+i\{(k_2\beta_n-\beta_{n-1})z^n\\&-(k_2-1)\beta_nz^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}+......+(\beta_{\mu+1}-\tau_2\beta_\mu)z^{\mu+1}+(\tau_2-1)\beta_\mu z^{\mu+1}\\&+(\beta_\mu-\beta_{\mu-1})z^\mu+......+(\beta_1-\beta_0)z\}.\end{aligned}$$

For $|z|=R$,we have, by using the hypothesis

$$|G(z)|\le|a_n|R^{n+1}+(k_1-1)|\alpha_n|R^n+(k_1\alpha_n-\alpha_{n-1})R^n+(\alpha_{n-1}-\alpha_{n-2})R^{n-1}+.......+(\alpha_{\lambda+1}-\tau_1\alpha_\lambda)R^{\lambda+1}$$

$$\begin{aligned}&+(1-\tau_1)|\alpha_\lambda|R^{\lambda+1}+|\alpha_\lambda-\alpha_{\lambda-1}|R^\lambda+......+|\alpha_1-\alpha_0|R+(k_2\beta_n-\beta_{n-1})R^n+(k_2-1)|\beta_n|R^n\\&+(\beta_{n-1}-\beta_{n-2})R^{n-1}+......+|\beta_{\mu+1}-\tau_2\beta_\mu|R^{\mu+1}+(1-\tau_2)|\beta_\mu|R^{\mu+1}+(\beta_\mu-\beta_{\mu-1})R^\mu\\&+......+|\beta_1-\beta_0|R\end{aligned}$$

$$\begin{aligned}&\le|a_n|R^{n+1}+R^n[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2}+......+(\alpha_{\lambda+1}-\tau_1\alpha_\lambda)\\&\quad+(1-\tau_1)|\alpha_\lambda|+|\alpha_\lambda-\alpha_{\lambda-1}|+......+|\alpha_1-\alpha_0|+|\alpha_0|-|\alpha_0|+k_2\beta_n-\beta_{n-1}+\beta_{n-1}\\&\quad-\beta_{n-1}+......+\beta_{\mu+1}-\tau_2\beta_\mu+|\beta_\mu-\beta_{\mu-1}|+......+|\beta_1-\beta_0|+|\beta_0|-|\beta_0|]\\&=|a_n|R^{n+1}+R^n[(k_1-1)|\alpha_n|+(k_2-1)|\beta_n|+k_1\alpha_n+k_2\beta_n+|\alpha_\lambda|+|\beta_\mu|+L+M\\&\quad-\tau_1(|\alpha_\lambda|+\alpha_\lambda)-\tau_2(|\beta_\mu|+\beta_\mu)-|\alpha_0|-|\beta_0|]\end{aligned}$$

for $R\ge1$

and for $R\le1$

$$|G(z)| \le |a_n|R^{n+1} + R[(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|].$$

Since G(z) is analytic for $|z| \le R, G(0) = 0$, it ,therefore, follows by Schwarz Lemma that

$$|G(z)| \le |a_n|R^{n+1} + R^n[(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|]$$

for $R \ge 1$
and

$$|G(z)| \le |a_n|R^{n+1} + R^n[(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|]$$

for $R \le 1$.
Therefore , for $|z| \le R$

$$|G(z)| \le [|a_n|R^{n+1} + R^n\{(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|\}]|z|$$

for $R \ge 1$
and

$$|G(z)| \le |a_n|R^{n+1} + R^n\{(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|\}]|z|$$

for $R \le 1$.
Hence , for $|z| \le R$

$$|F(z)| = |a_0 + G(z)|$$

$$\ge |a_0| - |G(z)|$$

$$\ge |a_0| - [\,|\mathrm{a}_\mathrm{n}|R^{n+1} + R^n\{(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|\}]|z|$$

$$> 0$$

if $|z| > \dfrac{|a_0|}{K}$, where

$$K = |\mathrm{a}_\mathrm{n}|R^{n+1} + R^n\{(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda) - \tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|\}$$

for $R \ge 1$
and

$$|F(z)| > 0$$

if $|z| > \dfrac{|a_0|}{K'}$ where

$$K' = |a_n|R^{n+1} + R^n\{(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$

$$-\tau_2(|\beta_\mu| + \beta_\mu) - |\alpha_0| - |\beta_0|\}]$$

for $R \le 1$.

This shows that F(z) and hence P(z) has all its zeros in $\frac{|a_0|}{K} \le |z|$ for $R \ge 1$ and in $\frac{|a_0|}{K'} \le |z|$ for $R \le 1$.

Again , for $|z| \le R$, it is easy to see that

$$|F(z)| \le |a_n|R^{n+1} + R^n[(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$

$$-\tau_2(|\beta_\mu| + \beta_\mu)]$$

$$= M$$

for $R \ge 1$

and

$$|F(z)| \le |a_n|R^{n+1} + R[(k_1-1)|\alpha_n| + (k_2-1)|\beta_n| + k_1\alpha_n + k_2\beta_n + |\alpha_\lambda| + |\beta_\mu| + L + M - \tau_1(|\alpha_\lambda| + \alpha_\lambda)$$

$$-\tau_2(|\beta_\mu| + \beta_\mu)] + (1-R)(|\alpha_0| + |\beta_0|)$$

$$= M'$$

for $R \le 1$.

Therefore, it follows, by using Lemma 2, that the number of zeros of F(z) and hence P(z) in $\frac{|a_0|}{K} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to $\frac{1}{\log c}\log\frac{M}{|a_0|}$ for $R \ge 1$

and the number of zeros of F(z) and hence P(z) in $\frac{|a_0|}{K'} \le |z| \le \frac{R}{c}, c > 1$ is less than or equal to

$\frac{1}{\log c}\log\frac{M'}{|a_0|}$ for $R \le 1$.

That completes the proof of Theorem 2.

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